7.3 Linear Systems of Equations. Gauss Elimination

Solution of Simultaneous Linear Equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

Two ways of solving them directly:

(1) elimination (explain later);

(2) determinants (Cramer's rule), which gives the solution as a ratio of two $n \times n$ determinants.

Gaussian Elimination

Forward Elimination

pivot → $2u + v + w = 1$ (1)

$4u + v = -2$ (2)

$-2u + 2v + w = 7$ (3)

Step 1: equation (2) + (– 2) x equation (1)

Step 2: equation (3) + (+1) x equation (1)

$2u + v + w = 1$ (4)

pivot → $-v - 2w = -4$ (5)

$3v + 2w = 8$ (6)

Step 3: equation (6) + (+3) x equation (5)

$2u + v + w = 1$ (7)

$-v - 2w = -4$ (8)

$-4w = -4$ (9)

Backward Substitution

$w = 1$

$v = 2$

$u = -1$

Elementary Transformation of Matrices – (i)

An elementary matrix of the first kind it an $n \times n$ diagonal matrix formed by replacing the ith diagonal element of identity matrix $\mathbf{I}$ with a nonzero constant q. For example, if $n = 4,\ i = 3$

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{Q}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/q & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det \mathbf{Q} = q$$

Elementary Transformation of Matrices – (ii)

An elementary matrix of the second kind is an $n \times n$ matrix $\mathbf{R}$ formed by interchanging anytwo rows i and j of the identity matrix $\mathbf{I}$. For example,

if $n = 4,\ i = 1$ and $j = 3$

$$\mathbf{R} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{R}^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det \mathbf{R} = -1$$

Elementary Transformation of Matrices – (iii)

An elementary matrix of the third kind it an $n \times n$ matrix $\mathbf{S}$ formed by inserting the a nonzero constant s into the off-diagonal position (i, j) of the identity matrix $\mathbf{I}$. For example, if $n = 4,\ i = 3$ and $j = 1$

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ s & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{S}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -s & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det \mathbf{S} = 1$$

Elementary Row Operation

Any row manipulation can be accomplished by pre-multiplication of elementary matrices!

$\mathbf{QA}_{n\times p}$: multiplication of all elements of the ith row in $\mathbf{A}$ by a constant q;

$\mathbf{RA}_{n\times p}$: interchange of the ith and jth row in $\mathbf{A}$;

$\mathbf{SA}_{n\times p}$: addition of a scalar multiple s of the jth row to the ith row.

Elementary Column Operation

Any column manipulation can be accomplished by post-multiplication of elementary matrices!

$\mathbf{A}_{p\times n}\mathbf{Q}$: multiplication of all elements of the ith column in $\mathbf{A}$ by a constant q;

$\mathbf{A}_{p\times n}\mathbf{R}$: interchange of the ith and jth column in $\mathbf{A}$;

$\mathbf{A}_{p\times n}\mathbf{S}$: addition of a scalar multiple s of the jth column to the ith column.

Gaussian Elimination = Triangular Factorization

$$\mathbf{Ax} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} = \mathbf{b}$$

$\Downarrow$ 3 elimination steps

$$\mathbf{Ux} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ -4 \end{bmatrix} = \mathbf{c}$$

upper triangular!!!

Step 1: 2nd equation + (-2) $\times$ 1st equation $\Rightarrow$ $\mathbf{S}_{21}\mathbf{A}$

where, $\mathbf{S}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Step 2: 3rd equation + (+1) $\times$ 1st equation $\Rightarrow$ $\mathbf{S}_{31}\mathbf{S}_{21}\mathbf{A}$

where, $\mathbf{S}_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Step 3: 3rd equation + (+3) $\times$ 2nd equation $\Rightarrow$ $\mathbf{S}_{32}\mathbf{S}_{31}\mathbf{S}_{21}\mathbf{A}$

where, $\mathbf{S}_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$

$$\mathbf{U} = \mathbf{S}_{32}\mathbf{S}_{31}\mathbf{S}_{21}\mathbf{A}$$

Upper triangular

Lower triangular

$$\mathbf{c} = \mathbf{S}_{32}\mathbf{S}_{31}\mathbf{S}_{21}\mathbf{b}$$

$$\text{Let } \hat{\mathbf{S}} = \mathbf{S}_{32}\mathbf{S}_{31}\mathbf{S}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\Rightarrow \mathbf{A} = \hat{\mathbf{S}}^{-1}\mathbf{U} = \mathbf{S}_{21}^{-1}\mathbf{S}_{31}^{-1}\mathbf{S}_{32}^{-1}\mathbf{U} = \mathbf{L}\mathbf{U}$$

$$\text{where } \mathbf{L} = \mathbf{S}_{21}^{-1}\mathbf{S}_{31}^{-1}\mathbf{S}_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix}$$

Note that 2, -1 and -3 are the negative values of the multipliers used in the 3 elimination steps.

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ e_{21} & 1 & 0 \\ e_{31} & e_{32} & 1 \end{bmatrix}$$

Thus, e_{ij} is the quantity that multiply row j when it is subtracted from row i to produce zero in the (i, j) entry.

Conclusion

If no pivots are zero, the matrix **A** can be written as a product **LU**. **L** is a lower triangular matrix with 1's on the main diagonal. **U** is an upper triangular matrix. The nonzero entries of **U** are the coefficients of the equations which appear after elimination and before back-substitution. The diagonal entries of **U** are the pivots.

Implications

Solve: $\mathbf{A}\mathbf{x}_n = \mathbf{b}_n$ n=1,2,3,$\cdots$

(1) Obtain $\mathbf{A} = \mathbf{L}\mathbf{U}$

(2) Solve $\mathbf{L}\mathbf{c}_n = \mathbf{b}_n$ with forward substitution for $\mathbf{c}_n$

(3) Solve $\mathbf{U}\mathbf{x}_n = \mathbf{c}_n$ with backward substitution for $\mathbf{x}_n$

Row Exchange

$$\mathbf{Ax} = \mathbf{b}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & d & 6 \\ 0 & c & 7 & 8 \end{bmatrix} \Rightarrow \mathbf{R}_{24}\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & c & 7 & 8 \\ 0 & 0 & d & 6 \\ 0 & 0 & 5 & 6 \end{bmatrix}$$

If $c = 0$, the problem is incurable and the matrix is called singular.

Elimination with Row Exchange

Assume $\mathbf{A}$ is nonsingular, then there exists a permutation matrix $\mathbf{R}$ that reorders the rows of $\mathbf{A}$ so that

$$\mathbf{RA} = \mathbf{LU}$$

Round Off Error

Consider

$$\mathbf{A} = \begin{bmatrix} 1.0 & 1.0 \\ 1.0 & 1.0001 \end{bmatrix}; \qquad \mathbf{A}' = \begin{bmatrix} 0.0001 & 1.0 \\ 1.0 & 1.0 \end{bmatrix}$$

First Point: Some matrices are extremely sensitive to small changes, and others are not. The matrix $\mathbf{A}$ is ill-conditioned (i.e. sensitive); $\mathbf{A}'$ is well-conditioned.

A is "nearly" singular

Singular matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \approx \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix}$$

$$(1)\ \mathbf{Ax} = \mathbf{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = 2 \\ x_2 = 0 \end{matrix}$$

$$(2)\ \mathbf{Ax}' = \mathbf{b}' = \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix} \Rightarrow \begin{matrix} x_1' = 1 \\ x_2' = 1 \end{matrix}$$

No numerical methods can provide this sensitivity to small perturbations!!!

Second Point: Even a well-conditioned matrix can be ruined by a poor algorithm.

$$\mathbf{A}'\mathbf{x} = \begin{bmatrix} 0.0001 & 1.0 \\ 1.0 & 1.0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 2.0 \end{bmatrix}$$

Correct solution:

$$x_1 = \frac{10000}{9999} \cong 1.00010001 \text{ (round off after 9th digit)}$$

$$x_2 = \frac{9998}{9999} \cong 0.99989998 \text{ (round off after 9th digit)}$$

If a calculator is capable of keeping only 3 digits, then Gaussian elimination gives the wrong answer!!!

$$(0.0001)x_1 + x_2 = 1 \qquad \text{(A)}$$

$$x_1 + x_2 = 2 \qquad \text{(B)}$$

Eq. (B) - 10000 × Eq.(A):

$$(1.0 - 0.0001 \times 10000.0)x_1 + (1.0 - 1.0 \times 10000.0)x_2$$

$$0 \qquad = 2.0 - 1.0 \times 10000.0$$

$$1.0 - 1.0 \times 10000.0 = -9999.0$$

$$= 1.00 - 1.00\text{E}4 \approx -1.00\text{E}4$$

$$2.0 - 1.0 \times 10000.0 = -9998.0$$

$$= 2.00 - 1.00\text{E}4 \approx -1.00\text{E}4$$

$x_2 \cong 1.00$ (not too bad)

Substituting into Eq.(A)

$x_1 = 0.00$ (This is wrong)

Third Point

A computer program should compare each pivot with all the other possible pivots in the same column. Choosing the largest of these candidates, and exchanging the corresponding rows so as to make this largest value pivot, is called *partial pivoting*.

Solution of m Equations with n Unknowns (m<n)

$$\mathbf{A}_{m\times n}\mathbf{x}=\mathbf{b}$$

pivot

$$\mathbf{A}=\begin{bmatrix}1 & 3 & 3 & 2\\ 2 & 6 & 9 & 5\\ -1 & -3 & 3 & 0\end{bmatrix}$$

$\Downarrow$ elementary row operation **S**

$$\begin{bmatrix}1 & 3 & 3 & 2\\ 0 & 0 & 3 & 1\\ 0 & 0 & 6 & 2\end{bmatrix}$$

pivot

$\Downarrow$ elementary row operation **S**

$$\mathbf{U}=\begin{bmatrix}1 & 3 & 3 & 2\\ 0 & 0 & 3 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Echelon Form

$$U = \begin{bmatrix} \circledast & * & * & * & * & * & * & * & * \\ 0 & \circledast & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \circledast & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \circledast \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fig. 2.2. The nonzero entries of a typical echelon matrix U.

Solution of m Equations with n Unknowns (m<n)

$$\mathbf{A}_{m\times n}\mathbf{x}=\mathbf{b}$$

pivot

$$\mathbf{A}=\begin{bmatrix}1 & 3 & 3 & 2\\ 2 & 6 & 9 & 5\\ -1 & -2 & 3 & 0\end{bmatrix}$$

$\Downarrow$ elementary row operation **S**

$$\begin{bmatrix}1 & 3 & 3 & 2\\ 0 & 0 & 3 & 1\\ 0 & 1 & 6 & 2\end{bmatrix}$$

$\Downarrow$ elementary row operation **R**

$$\mathbf{U}=\begin{bmatrix}1 & 3 & 3 & 2\\ 0 & 1 & 6 & 2\\ 0 & 0 & 3 & 1\end{bmatrix}$$

CONCLUSION

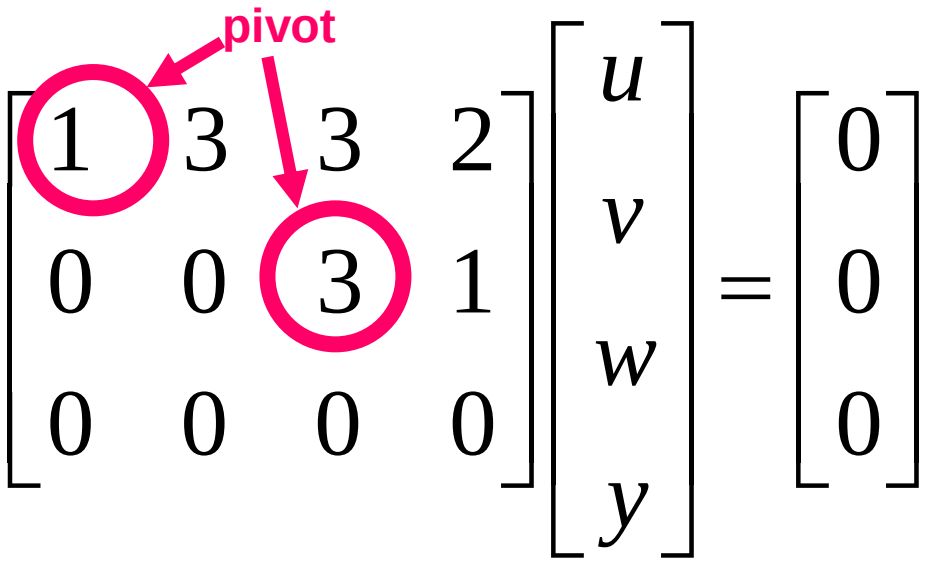
To any m by n matrix $\mathbf{A}$ there correspond a permutation matrix $\mathbf{R}$, a lower triangular matrix $\mathbf{L}$ with unit diagonal, and an m by n echelon matrix $\mathbf{U}$, such that

$$\mathbf{RA} = \mathbf{LU}$$

Homogeneous Solution

$$\mathbf{b} = \mathbf{0}$$

$$\mathbf{Ax} = \mathbf{0} \quad \Rightarrow \quad \mathbf{Ux} = \mathbf{0}$$



u, w : basic variables

v, y : free variables

$\Rightarrow$ Express basic variables in terms of free variables!

$$3w+y=0 \;\rightarrow\; w = -\frac{1}{3}y$$

$$u + 3v + 3w + 2y = 0 \;\rightarrow\; u = -3v - y$$

$$\mathbf{x} = \begin{bmatrix} -3v - y \\ v \\ -y/3 \\ y \end{bmatrix} = v\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} -1 \\ 0 \\ -1/3 \\ 1 \end{bmatrix}$$

- All solutions are linear combinations of

$$\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 0 \\ -1/3 \\ 1 \end{bmatrix}$$

- Within the 4-D space of all possible **x**, the solution of **Ax = 0** form a 2-D subspace.

$\Rightarrow$ the nullspace of **A**

Conclusions

- Every homogeneous system $\mathbf{Ax=0}$, if it has more unknowns than equations ($n>m$), has infinitely many nontrivial solutions.
- The nullspace is a subspace of the same "dimension" as the number of *free variables.*
- The nullspace is a subspace of R^n.

Subspace

A subspace of a vector space is a subset that satisfies two requirements:

1. If we add any two vectors $\mathbf{x}$ and $\mathbf{y}$ in the subspace, the sum $\mathbf{x}+\mathbf{y}$ is still in the subspace.
2. If we multiply any vector $\mathbf{x}$ in the subspace by any scalar c, the multiple $c\mathbf{x}$ is still in the subspace.

Note that the zero vector belongs to every subspace.

Inhomogeneous Solution

$\mathbf{b} \neq \mathbf{0}$

$\mathbf{Ax} = \mathbf{LUx} = \mathbf{b}$

$\mathbf{Ux} = \mathbf{L}^{-1}\mathbf{b} = \mathbf{c}$

$$\mathbf{Ux} = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 5b_1 \end{bmatrix} = \mathbf{c}$$

Note that the equations are inconsistent unless

$$b_3 - 2b_2 + 5b_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = 0$$

In other words, the set of attainable vectors **b** is not the whole of the 3-D space.

$$\mathbf{Ax} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} \mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$u\mathbf{a}_1 + v\mathbf{a}_2 + w\mathbf{a}_3 + y\mathbf{a}_4 = \mathbf{b}$$

$$u \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + v \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix} + w \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Conclusion

The system $\mathbf{Ax=b}$ is solvable if and only if the vector $\mathbf{b}$ can be expressed as a linear combination of the columns of $\mathbf{A}$.

Note that

$$\mathbf{a}_2 = 3\mathbf{a}_1$$

$$\mathbf{a}_4 = \mathbf{a}_1 + \frac{1}{3}\mathbf{a}_3$$

$$\mathbf{b} = u'\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + w'\begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix} = u'\mathbf{a}_1 + w'\mathbf{a}_3$$

Conclusions

- $\mathbf{Ax} = \mathbf{b}$ can be solved iff $\mathbf{b}$ lies in the plane that is spanned by $\mathbf{a}_1$ and $\mathbf{a}_3$.
- The plane is a subspace of R^m called column space of the matix **A**.
- The equation **Ax = b** can be solved iff **b** lies in the column space.

$$\mathbf{Ax} = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix} = \mathbf{b}$$

$$\mathbf{Ux} = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \mathbf{c}$$

$$3w + y = 3 \quad \rightarrow \quad w = 1 - \frac{1}{3} y$$

$$u + 3v + 3w + 2y = 1 \quad \rightarrow \quad u = -2 - y - 3v$$

$$\mathbf{x} = \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 0 \\ -1/3 \\ 1 \end{bmatrix}$$

particular soln

homogeneous solution
Ax=0

CONCLUSIONS

- Suppose the mxn matrix $\mathbf{A}$ is reduced by elementary operations and row exchanges to a matrix $\mathbf{U}$ in echelon form.
- Let there be r nonzero pivots; the last $m-r$ rows of $\mathbf{U}$ are zero. Then there will be r basic variables and $n-r$ free variables, corresponding to the columns of $\mathbf{U}$ with and without pivots respectively.

CONCLUSIONS

- The nullspace, formed of solutions to $\mathbf{Ax=0}$, has the $n-r$ free variables as the independent parameters. If $r=n$, there are no free variables and the null space contains only $\mathbf{x=0}$.
- Solutions exist for every right side $\mathbf{b}$ iff $r=m$, then $\mathbf{U}$ has no zero rows, and $\mathbf{Ux=c}$ can be solved by back-substitution.

CONCLUSIONS

- If $r<m$, $\mathbf{U}$ will have $m-r$ zero rows and there are m-r constraints on $\mathbf{b}$ in order for $\mathbf{Ax=b}$ to be solvable. If one particular solution exists, then every other solution differs from it by a vector in the nullspace of $\mathbf{A}$.
- The number r is called the rank of the matrix $\mathbf{A}$.

7.5 Solutions of Linear Systems: Existence, Uniqueness

Theorem 1

Fundamental Theorem for Linear Systems

(a) Existence.

A linear system of m equations in n unknowns $x_1, \dots, x_n$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m .
\end{aligned}
\tag{1}
$$

is **consistent**, *that is, has solutions, if and only if the coefficient matrix* $\mathbf{A}$ *and the augmented matrix* $\tilde{\mathbf{A}}$ *have the same rank.*

Theorem 1 (continued)

Fundamental Theorem for Linear Systems (continued)

(a) Existence. (continued)

Here,

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \quad \textit{and} \quad \tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

Theorem 1 (continued)

Fundamental Theorem for Linear Systems (continued)

(b) Uniqueness.

The system (1) *has precisely one solution if and only if this common rank r of* $\mathbf{A}$ *and* $\tilde{\mathbf{A}}$ *equals n.*

Theorem 1 (continued)

Fundamental Theorem for Linear Systems (continued)

(c) Infinitely many solutions.
If this common rank r is less than n, the system (1) *has infinitely many solutions. All of these solutions are obtained by determining r suitable unknowns* (*whose submatrix of coefficients must have rank r*) *in terms of the remaining n − r unknowns, to which arbitrary values can be assigned.*

Theorem 1 (continued)

Fundamental Theorem for Linear Systems (continued)

(d) Gauss elimination (Sec. 7.3).

If solutions exist, they can all be obtained by the Gauss elimination. (This method will automatically reveal whether or not solutions exist)

Homogeneous Linear System

A linear system (1) is called **homogeneous** if all the b_j's are zero, and **nonhomogeneous** if one or several b_j's are not zero.

Theorem 2

Homogeneous Linear System

A homogeneous linear system

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}
\tag{4}
$$

always has the **trivial solution** $x_1 = 0, \ldots, x_n = 0.$

Homogeneous Linear System

Theorem 2 (continued)

Homogeneous Linear System (continued)

Nontrivial solutions exist if and only if rank $\mathbf{A} < n$. *If* rank $\mathbf{A} = r < n$, *these solutions, together with* $\mathbf{x} = \mathbf{0}$, *form a vector space of dimension* $n - r$ called the **solution space** of (4).

In particular, if $\mathbf{x}_{(1)}$ *and* $\mathbf{x}_{(2)}$ *are solution vectors of* (4), *then* $\mathbf{x} = c_1 \mathbf{x}_{(1)} + c_2 \mathbf{x}_{(2)}$ *with any scalars* c_1 *and* c_2 *is a solution vector of* (4). (This ***does not hold*** for nonhomogeneous systems. Also, the term *solution space is used for homogeneous* systems only.)

The solution space of (4) is also called the **null space** of **A** because $\mathbf{Ax} = \mathbf{0}$ for every **x** in the solution space of (4). Its dimension is called the **nullity** of **A**. Hence Theorem 2 states that

(5) $\qquad \text{rank } \mathbf{A} + \text{nullity } \mathbf{A} = n$

where n is the number of unknowns (number of columns of **A**).

By the definition of rank we have rank $\mathbf{A} \leq m$ in (4). Hence if $m < n$, then rank $\mathbf{A} < n$.

Theorem 3

Homogeneous Linear System with Fewer Equations Than Unknowns

A homogeneous linear system with fewer equations than unknowns always has nontrivial solutions.

Nonhomogeneous Linear System

Theorem 4

Nonhomogeneous Linear System

If a nonhomogeneous linear system (1) *is consistent, then all of its solutions are obtained as*

(6) $$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_h$$

where $\mathbf{x}_0$ *is any* (fixed) *solution of* (1) *and* $\mathbf{x}_h$ *runs through all the solutions of the corresponding homogeneous system* (4).

7.7 Determinants. Cramer's Rule

A **determinant of order** n is a scalar associated with an $n \times n$ (hence ***square!***) matrix $\mathbf{A} = [a_{jk}]$, and is denoted by

(1)
$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}.$$

For $n = 1$, this determinant is defined by

(2)
$$D = a_{11}.$$

For $n \geq 2$ by

(3a) $D = a_{j1}C_{j1} + a_{j2}C_{j2} + \ldots + a_{jn}C_{jn}$ ($j = 1, 2, \ldots,$ or n)

or

(3b) $D = a_{1k}C_{1k} + a_{2k}C_{2k} + \ldots + a_{nk}C_{nk}$ ($k = 1, 2, \ldots,$ or n).

Here,

$$C_{jk} = (-1)^{j+k}M_{jk}$$

and M_{jk} is a determinant of order $n - 1$, namely, the determinant of the submatrix of **A** obtained from **A** by omitting the row and column of the entry a_{jk}, that is, the jth row and the kth column.

In this way, D is defined in terms of n determinants of order $n-1$, each of which is, in turn, defined in terms of $n-1$ determinants of order $n-2$ and so on—until we finally arrive at second-order determinants, in which those submatrices consist of single entries whose determinant is defined to be the entry itself.

From the definition it follows that *we may* ***expand*** *D by any row or column,* that is, choose in (3) the entries in any row or column, similarly when expanding the C_{jk}'s in (3), and so on.

This definition is unambiguous, that is, it yields the same value for D no matter which columns or rows we choose in expanding. A proof is given in App. 4.

Terms used in connection with determinants are taken from matrices. In D we have n^2 **entries** a_{jk} also n **rows** and n **columns**, and a **main diagonal** on which $a_{11}, a_{22}, \cdots, a_{nn}$ stand.

Two terms are new: M_{jk} is called the **minor** *of* a_{jk} *in* D, and C_{jk} the **cofactor** *of* a_{jk} *in* D.

For later use we note that (3) may also be written in terms of minors

(4a) $$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \qquad (j = 1, 2, \cdots, \text{ or } n)$$

(4b) $$D = \sum_{j=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \qquad (k = 1, 2, \cdots, \text{ or } n)$$

EXAMPLE 1

Minors and Cofactors of a Third-Order Determinant

In (4) of the previous section the minors and cofactors of the entries in the first column can be seen directly. For the entries in the second row the minors are

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \quad M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

and the cofactors are $C_{21} = -M_{21}$, $C_{22} = +M_{22}$, and $C_{23} = -M_{23}$ Similarly for the third row—write these down yourself. And verify that the signs in C_{jk} form a **checkerboard pattern**

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

EXAMPLE 2

Expansions of a Third-Order Determinant

$$D = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix}$$

$$= 1(12 - 0) - 3(4 + 4) + 0(0 + 6) = -12.$$

This is the expansion by the first row. The expansion by the third column is

$$D = 0\begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix} - 4\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 0 - 12 + 0 = -12.$$

Verify that the other four expansions also give the value −12.

General Properties of Determinants

- There is an attractive way of finding determinants (1) that consists of applying elementary row operations to (1).
- By doing so we obtain an "upper triangular" determinant (see Sec. 7.1, for definition with "matrix" replaced by "determinant") whose value is then very easy to compute, being just the product of its diagonal entries.
- This approach is ***similar*** (***but not the same*!**) to what we did to matrices in Sec. 7.3. *In particular, be aware that interchanging two rows in a determinant introduces a multiplicative factor of* −1 *to the value of the determinant*! Details are as follows.

THEOREM 1

Behavior of an *n*th-Order Determinant under Elementary Row Operations

(a) *Interchange of two rows multiplies the value of the determinant by* −1.

(b) *Addition of a multiple of a row to another row does not alter the value of the determinant.*

(c) *Multiplication of a row by a nonzero constant c multiplies the value of the determinant by c.* (This holds also when $c = 0$, but no longer gives an elementary row operation.)

EXAMPLE 4

Evaluation of Determinants
by Reduction to Triangular Form

Because of Theorem 1 we may evaluate determinants by reduction to triangular form, as in the Gauss elimination for a matrix. For instance (with the blue explanations always referring to the ***preceding determinant***)

$$D = \begin{vmatrix} 2 & 0 & -4 & 6 \\ 4 & 5 & 1 & 0 \\ 0 & 2 & 6 & -1 \\ -3 & 8 & 9 & 1 \end{vmatrix}$$

EXAMPLE 4 (continued)

Evaluation of Determinants by Reduction to Triangular Form (continued)

$$= \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 2 & 6 & -1 \\ 0 & 8 & 3 & 10 \end{vmatrix} \quad \begin{matrix} \\ \text{Row 2} - 2\,\text{Row 1} \\ \\ \text{Row 4} + 1.5\,\text{Row 1} \end{matrix}$$

$$= \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & -11.4 & 29.2 \end{vmatrix} \quad \begin{matrix} \\ \\ \text{Row 3} - 0.4\,\text{Row 2} \\ \text{Row 4} - 1.6\,\text{Row 2} \end{matrix}$$

EXAMPLE 4 (continued)

Evaluation of Determinants by Reduction to Triangular Form (continued)

$$= \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & -0 & 47.25 \end{vmatrix} \quad \text{Row 4 + 4.75 Row 3}$$

$$= 2 \cdot 5 \cdot 2.4 \cdot 47.25 = 1134.$$

THEOREM 2

Further Properties of *n*th-Order Determinants

(a)–(c) *in Theorem* 1 *hold also for columns.*

(d) ***Transposition*** *leaves the value of a determinant unaltered.*

(e) ***A zero row or column*** *renders the value of a determinant zero.*

(f) ***Proportional rows or columns*** *render the value of a determinant zero. In particular, a determinant with two identical rows or columns has the value zero.*

THEOREM 3

Rank in Terms of Determinants

Consider an $m \times n$ *matrix* $\mathbf{A} = [a_{jk}]$:

(1) $\mathbf{A}$ *has rank* $r \geq 1$ *if and only if* $\mathbf{A}$ *has an* $r \times r$ *submatrix with a nonzero determinant.*

(2) *The determinant of any square submatrix with more than* r *rows, contained in* $\mathbf{A}$ *(if such a matrix exists!) has a value equal to zero.*

Furthermore, if $m = n$, *we have:*

(3) *An* $n \times n$ *square matrix* $\mathbf{A}$ *has rank* n *if and only if*

$$\det \mathbf{A} \neq 0.$$

Cramer's Rule

THEOREM 4

Cramer's Theorem
(Solution of Linear Systems by Determinants)

(a) *If a linear system of n equations in the same number of unknowns* $x_1, \dots, x_n$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\cdots\cdots\cdots\cdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
\tag{6}
$$

has a nonzero coefficient determinant $D = \det \mathbf{A}$, *the system has precisely one solution.*

THEOREM 4 (continued)

Cramer's Theorem (Solution of Linear Systems by Determinants) (continued)

This solution is given by the formulas

$$\textbf{(7)}\quad x_1 = \frac{D_1}{D}, \qquad x_2 = \frac{D_2}{D}, \quad \cdots \quad , \quad x_n = \frac{D_n}{D} \qquad \textbf{(Cramer's rule)}$$

where D_k is the determinant obtained from D by replacing in D the kth column by the column with the entries $b_1, \ldots, b_n$.

(b) *Hence if the system* (6) *is* ***homogeneous*** *and* $D \neq 0$*, it has only the trivial solution* $x_1 = 0, x_2 = 0, \ldots, x_n = 0$.

If $D = 0$ *the homogeneous system also has nontrivial solutions*.

7.8 Inverse of a Matrix. Gauss–Jordan Elimination

In this section we consider* square *matrices exclusively.

The **inverse** of an $n \times n$ matrix $\mathbf{A} = [a_{jk}]$ is denoted by $\mathbf{A}^{-1}$ and is an $n \times n$ matrix such that

(1) $$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

where $\mathbf{I}$ is the $n \times n$ unit matrix (see Sec. 7.2).

If $\mathbf{A}$ has an inverse, then $\mathbf{A}$ is called a **nonsingular matrix**. If $\mathbf{A}$ has no inverse, then $\mathbf{A}$ is called a **singular matrix**.

If $\mathbf{A}$ *has an inverse, the inverse is unique.*

Indeed, if both $\mathbf{B}$ and $\mathbf{C}$ are inverses of $\mathbf{A}$, then $\mathbf{AB} = \mathbf{I}$ and $\mathbf{CA} = \mathbf{I}$ so that we obtain the uniqueness from

$$\mathbf{B} = \mathbf{IB} = (\mathbf{CA})\mathbf{B} = \mathbf{C}(\mathbf{AB}) = \mathbf{CI} = \mathbf{C}.$$

THEOREM 1

Existence of the Inverse

The inverse $\mathbf{A}^{-1}$ *of an* $n \times n$ *matrix* $\mathbf{A}$ *exists if and only if* rank $\mathbf{A} = n$, *thus* (by Theorem 3, Sec. 7.7) *if and only if* $\det \mathbf{A} \neq 0$. *Hence* $\mathbf{A}$ *is nonsingular if* rank $\mathbf{A} = n$ *and is singular if* rank $\mathbf{A} < n$.

Determination of the Inverse by the Gauss–Jordan Method

EXAMPLE 4 Finding the Inverse of a Matrix by Gauss–Jordan Elimination

Determine the inverse $\mathbf{A}^{-1}$ of

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

EXAMPLE 4 (continued) Finding the Inverse of a Matrix by Gauss–Jordan Elimination

Solution.

We apply the Gauss elimination (Sec. 7.3) to the following $n \times 2n = 3 \times 6$ matrix, where **BLUE** always refers to the previous matrix.

$$[\mathbf{A} \quad \mathbf{I}] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \quad \begin{array}{l} \\ \text{Row 2} + 3\,\text{Row 1} \\ \text{Row 3} - \text{Row 1} \end{array}$$

EXAMPLE 4 (continued) Finding the Inverse of a Matrix by Gauss–Jordan Elimination

Solution. *(continued 1)*

$$= \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right] \quad \text{Row 3} - \text{Row 2}$$

EXAMPLE 4 (continued) Finding the Inverse of a Matrix by Gauss–Jordan Elimination

Solution. *(continued 2)*

This is [**U** **H**] as produced by the Gauss elimination. Now follow the additional Gauss–Jordan steps, reducing **U** to **I**, that is, to diagonal form with entries 1 on the main diagonal.

$$
\begin{bmatrix} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 3.5 & 1.5 & 0.5 & 0 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{bmatrix}
$$

$$
= \qquad \begin{matrix} - \text{ Row 1} \\ 0.5 \text{ Row 2} \\ -0.2 \text{ Row 3} \end{matrix}
$$

EXAMPLE 4 (continued) Finding the Inverse of a Matrix by Gauss–Jordan Elimination

Solution. *(continued 3)*

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0.6 & 0.4 & -0.4 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] \quad \begin{array}{l} \text{Row 1} + 2\ \text{Row 3} \\ \text{Row 2} - 3.5\ \text{Row 3} \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] \quad \text{Row 1} + \text{Row 2}$$

EXAMPLE 4 (continued) Finding the Inverse of a Matrix by Gauss–Jordan Elimination

Solution. *(continued 4)*

The last three columns constitute $\mathbf{A}^{-1}$. Check:

$$\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. Similarly $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Formulas for Inverses

THEOREM 2

Inverse of a Matrix by Determinants

The inverse of a nonsingular $n \times n$ matrix $\mathbf{A} = [a_{jk}]$ is given by

$$(4) \quad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \left[C_{jk} \right]^{\mathsf{T}} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{21} & C_{22} & \cdots & C_{n2} \\ \cdot & \cdot & \cdots & \cdot \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix},$$

where C_{jk} is the cofactor of a_{jk} in $\det \mathbf{A}$ (see Sec. 7.7).

THEOREM 2 (continued)

Inverse of a Matrix by Determinants (continued)

(**CAUTION**! Note well that in $\mathbf{A}^{-1}$, the cofactor C_{jk} occupies the same place as a_{kj} (not a_{jk}) does in $\mathbf{A}$.)

In particular, the inverse of

$$(4^*) \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \textit{is} \quad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

EXAMPLE 2 Inverse of a 2×2 Matrix by Determinants

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}, \qquad \mathbf{A}^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$$

EXAMPLE 3 Further Illustration of Theorem 2

Using (4), find the inverse of

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

EXAMPLE 3 (continued) Further Illustration of Theorem 2

Solution. We obtain $\det \mathbf{A} = -1(-7) - 1 \cdot 13 + 2 \cdot 8 = 10$, and in (4),

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7, \quad C_{21} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2, \quad C_{31} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3,$$

$$C_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13, \quad C_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2, \quad C_{32} = -\begin{vmatrix} -1 & 2 \\ 3 & -1 \end{vmatrix} = 7,$$

$$C_{13} = \begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix} = 8, \quad C_{23} = -\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = 2, \quad C_{33} = \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = -2,$$

EXAMPLE 3 (continued) Further Illustration of Theorem 2

Solution. **(continued)**

so that by (4), in agreement with Example 1,

$$\mathbf{A}^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}.$$

Products can be inverted by taking the inverse of each factor and multiplying these inverses ***in reverse order,***

(7) $$(\mathbf{AC})^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1}.$$

Hence for more than two factors,

(8) $$(\mathbf{AC} \dots \mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1} \dots \mathbf{C}^{-1}\mathbf{A}^{-1}.$$

Unusual Properties of Matrix Multiplication. Cancellation Laws

[1] Matrix multiplication is not commutative, that is, in general we have

$$\mathbf{AB} \neq \mathbf{BA}.$$

[2] $\mathbf{AB} = \mathbf{0}$ does not generally imply $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$ (or $\mathbf{BA} = \mathbf{0}$); for example,

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

[3] $\mathbf{AC} = \mathbf{AD}$ does not generally imply $\mathbf{C} = \mathbf{D}$ (even when $\mathbf{A} \neq \mathbf{0}$).

THEOREM 3

Cancellation Laws

Let **A**, **B**, **C** *be* $n \times n$ *matrices. Then:*

(a) *If* rank **A** = n *and* **AB** = **AC**, *then* **B** = **C**.

(b) *If* rank **A** = n, *then* **AB** = **0** *implies* **B** = **0**. *Hence if* **AB** = **0**, *but* **A** ≠ **0** *as well as* **B** ≠ **0**, *then* rank **A** < n *and* rank **B** < n.

(c) *If* **A** *is singular, so are* **BA** *and* **AB**.

Determinants of Matrix Products

THEOREM 4

Determinant of a Product of Matrices

For any $n \times n$ matrices $\mathbf{A}$ *and* $\mathbf{B}$,

(10) $$\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det \mathbf{A} \det \mathbf{B}.$$